

Classification of the Entangled states $L \times N \times N$

Jun-Li Li¹, Shi-Yuan Li², and Cong-Feng Qiao^{1,3,*}

¹Department of Physics, Graduate University of Chinese Academy of Sciences
YuQuan Road 19A, Beijing 100049, China

²School of Physics, Shandong University, 250100, China

³Theoretical Physics Center for Science Facilities (TPCSF), CAS
YuQuan Road 19B, Beijing 100049, China

Abstract

We presented a general classification scheme for the tripartite $L \times N \times N$ entangled system under stochastic local operation and classical communication. The whole classification procedure consists of two correlated parts: the simultaneous similarity transformation of a commuting matrix pair into a canonical form and the study of internal symmetry of parameters in the canonical form. Based on this scheme, a concrete example of entanglement classification for a $3 \times N \times N$ system is given.

PACS numbers: 03.67.Mn, 03.65.Ud, 02.10.Xm

1 Introduction

Entanglement represents the essence of quantum theory compared to the classical theories. With the development of quantum information science, entanglement is now expected to be generated from a wide range of systems and is generally regarded as the key physical resource in implementing various quantum information tasks [1]. The classification of entanglement is generally a mathematically difficult problem. It has been

*Corresponding author: qiaocf@gucas.ac.cn

pointed out that only in a kind of tripartite system, which has a qubit as one of the subsystems ($2 \times M \times N$), one may find finite number of entanglement classes [2]. The complete entanglement classification of a $2 \times M \times N$ pure system under stochastic local operation and classical communication (SLOCC) has been done in Refs. [3, 4], where canonical forms are constructed via Jordan decomposition of the matrices. There the eigenvalues of the Jordan forms serve as the continuous parameters in the entanglement classes [3]. Despite the great progress made in $2 \times M \times N$ systems, in many classification schemes the classification of $3 \times N \times N$ turns out to be a “wild” problem [5]. Attempts toward the classification of a multipartite system in literature mainly concentrate on its tensor ranks or local ranks of the tensor form of the quantum state, and there is still no systematic method for constructing the canonical form of $L \times N \times N$ under SLOCC in literature to our knowledge. Recently, a method featured by the decomposition of the entanglement classification under local unitary transformations into different parts has been proposed [6]. In this work, by decomposing entanglement classification into two complementary ingredients, we propose an entanglement classification scheme for a tripartite $L \times N \times N$ system under SLOCC.

The structure of the paper is as follows. In Sec. 2, by representing the $L \times N \times N$ quantum state in the form of a three-way tensor, its entanglement classification under SLOCC is transformed into two correlated problems: the simultaneous similarity transformation of a commuting matrix pair into a canonical form and the representation of the symmetry of parameters in the canonical form. In Sec. 3, a concrete example of entanglement classification for a $3 \times N \times N$ system is given in which the explicit form of the symmetry property of the state is shown. A brief summary and concluding remarks are given in Sec. 4.

2 The classification scheme

Considering a general tripartite pure quantum state with dimensions $L \times N \times N$, the wave function can be written as

$$\psi = \sum_{i,j,k=1}^{L,N,N} \gamma_{ijk} |i\rangle |j\rangle |k\rangle, \quad (1)$$

where $|i\rangle, |j\rangle, |k\rangle$ are bases of the three parts respectively. The complex coefficients γ_{ijk} can be treated as a three-order tensor which then turns into the form of a tuple of L complex $N \times N$ matrices, i.e.,

$$\psi = (\Gamma_1, \Gamma_2, \dots, \Gamma_L), \quad (2)$$

where $\Gamma_i, i \in \{1, 2, \dots, L\}$ are $N \times N$ complex matrices and have $[\gamma_i]_{jk}$ as their elements. A state $\psi' = (\Gamma'_1, \Gamma'_2, \dots, \Gamma'_L)$ is said to be SLOCC equivalent [2] to ψ in the case of

$$\sum_{j=1}^L T_{ij} P \Gamma_j Q = \Gamma'_i, \quad i = \{1, 2, \dots, L\}. \quad (3)$$

Here, T (where its elements are T_{ij}) is an $L \times L$ invertible matrix, i.e., invertible local operator (ILO) acting on the first partite, and P and Q are $N \times N$ ILOs acting on the second and third partite, respectively.

For a tuple of matrices $(\Gamma_1, \Gamma_2, \dots, \Gamma_L)$, there always exists a tuple of numbers (t_1, t_2, \dots, t_L) $t_i \in \mathbb{C}$, such that the linear combination of Γ_i s,

$$\Gamma = \sum_{j=1}^L t_j \Gamma_j \quad (4)$$

gives the maximum rank, the $r(\Gamma)$. (We denote the rank of a matrix as $r(\cdot)$ hereafter.) For every quantum state in the form of Eq.(2), we can always transform them into the form where the Γ_1 has the maximum rank by performing the transformation T , of which the first row is chosen to be the tuple of (t_1, t_2, \dots, t_L) , i.e., $T_{1j} = t_j$. Therefore, in the following we can then restrict our discussion of the quantum state classification to that of the tuple $(\Gamma_1, \Gamma_2, \dots, \Gamma_L)$, where Γ_1 has possessed the maximum rank.

The strategy we adopt in the entanglement classification in this work is quite transparent, which can be roughly outlined as the manipulation of three commutative ILOs T, P, Q in different order and combination. We first transform the quantum state in a tuple of matrices according to specific rank property for each matrix by a subset of ILO T , then obtain a canonical form by using ILOs P, Q , and finally achieve the entanglement classification by reconsidering the rest a subset of ILO T as a degeneracy condition between the canonical forms. Following, we explain in detail the classification procedures, in which two situations of matrix Γ_1 with full and nonfull ranks are discussed separately.

2.1 The case of Γ_1 with full rank

In this case, obviously every quantum state in form (2) can be transformed into the following form by ILOs P, Q defined in Eq.(3)

$$\psi = (E, \Gamma_2, \dots, \Gamma_L), \quad (5)$$

where E is a unit matrix. Then a subsequent ILO transformation

$$T = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ -t_{21} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -t_{L1} & 0 & \cdots & 1 \end{pmatrix}, \quad (6)$$

where t_{i1} is one of the eigenvalues of Γ_i , can always be applied to ψ and makes $r(\Gamma_i) < N$ after the transformation. By definition, two such quantum states ψ, ψ' are equivalent under the transformations of ILOs P, Q if and only if

$$(PEQ, P\Gamma_2Q, \dots, P\Gamma_LQ) = (E, \Gamma'_2, \dots, \Gamma'_L), \quad (7)$$

which tells $Q = P^{-1}$. Equation (7) now turns into a simultaneous similarity relation of matrices $(\Gamma_2, \dots, \Gamma_L)$,

$$(PEP^{-1}, P\Gamma_2P^{-1}, \dots, P\Gamma_LP^{-1}) = (E, \Gamma'_2, \dots, \Gamma'_L). \quad (8)$$

According to [7, 8], all these kinds of problems can be transformed into the simultaneous similarity of a pair of commuting matrices, i.e., Eq.(8) is equivalent to

$$(E, PA_2P^{-1}, PA_3P^{-1}) = (E, A'_2, A'_3), \quad (9)$$

where $A_{2,3}$ are composed by Γ_i , and $[A_2, A_3] = 0$. This makes the entanglement classification of a $3 \times N \times N$ system of great importance. The canonical form of Eq.(9) under the similarity transformation can be written as $\{E, J, A\}$, where J is the Jordan form of A_2 , A is the canonical form of A_3 , and $[A, J] = 0$. J can be represented as $J = \bigoplus J_{n_i}(\lambda_i)$, where $J_{n_i}(\lambda_i)$ are $n_i \times n_i$ matrices which take the following form:

$$\begin{pmatrix} \lambda_i & 1 & 0 & \cdots & 0 \\ 0 & \lambda_i & 1 & \cdots & 0 \\ 0 & 0 & \lambda_i & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & \cdots & \lambda_i \end{pmatrix}. \quad (10)$$

Due to $[A, J] = 0$, the matrix A generally has the form of $A = \bigoplus A_i$ accordingly, where A_i is a $n_i \times n_i$ triangular Toeplitz matrix (see Appendix A):

$$A_i = \begin{pmatrix} a_{i0} & a_{i1} & a_{i2} & a_{i3} & \cdots & a_{in_i-1} \\ 0 & a_{i0} & a_{i1} & a_{i2} & \cdots & a_{in_i-2} \\ 0 & 0 & a_{i0} & a_{i1} & \cdots & a_{in_i-3} \\ 0 & 0 & 0 & a_{i0} & \cdots & a_{in_i-4} \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_{i0} \end{pmatrix}. \quad (11)$$

Since the Jordan block can be defined as a polynomial function of $J_{n_i}(0)$,

$$J_{n_i}(\lambda_i) = \lambda_i + J_{n_i}(0), \quad (12)$$

the matrix A_i can also be similarly defined as

$$A_i = \sum_{n=0}^{n_i-1} a_{in} J_{n_i}^n(0). \quad (13)$$

By defining $f_{[A_i]}(x) = \sum_{n=0}^{n_i-1} a_{in} x^n$, we have $A_i = f_{[A_i]}(J_{n_i}(0))$.

However, there still exist other transformation T s that can relate $\psi = (E, \Gamma_2, \dots, \Gamma_L)$ to another quantum state $\psi' = (E, \Gamma'_2, \dots, \Gamma'_L)$. The effect of the transformation T is introducing a degeneracy on the canonical form of (E, J, A) . That is, the states (E, J, A) and (E, J', A') with parametric difference may belong to the same entanglement class. Define a set $\vec{\mathbf{a}} = \{\lambda_i, a_{in} | 0 \leq n \leq n_i - 1\}$, which contains all the parameters in (E, J, A) .

The degeneracy can then be expressed as a symmetry transformation \mathcal{P} on the parameters of the canonical form, i.e.,

$$\vec{\mathbf{a}}' = \mathcal{P} \vec{\mathbf{a}}. \quad (14)$$

Thus, the canonical form is of the entanglement classification of the quantum state up to the symmetry \mathcal{P} .

2.2 The case of Γ_1 with non-full rank

In this case, all of the quantum states can be transformed into the following form by ILOs P, Q

$$\psi = (\Lambda, \Gamma_2, \dots, \Gamma_L). \quad (15)$$

Here, Λ has the following form:

$$\Lambda = \begin{pmatrix} E_{(N-i) \times (N-i)} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{i \times i} \end{pmatrix} = \left(\begin{array}{ccc|ccc} 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 & \cdots & 0 \\ \hline 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{array} \right) \quad (16)$$

with $i > 0$.

Then we apply another type of ILOs, P_Λ, Q_Λ , such that they keep Λ invariant and transform all the Γ_j , $j \in \{2, \dots, L\}$ to a form of direct sum of two matrices. The philosophy here is the same as that of the less than full rank case in [3], that is,

$$\Gamma_j = \left(\begin{array}{ccc|ccc} \gamma_{j11} & \cdots & \gamma_{j1n} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \gamma_{jn1} & \cdots & \gamma_{jnn} & 0 & \cdots & 0 \\ \hline 0 & \cdots & 0 & \beta_{j11} & \cdots & \beta_{j1m} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \beta_{jm1} & \cdots & \beta_{jmm} \end{array} \right). \quad (17)$$

Here, $N = n + m$, and the matrices β_j cannot be simultaneously 0 for j and cannot be further decomposed into the direct sum of submatrices via P_Λ, Q_Λ . From this and from

the condition that the first matrix Λ has the maximum rank in comparison to Γ_j , we can also deduce $m > i$. With the partition (17), Λ can be partitioned accordingly as

$$\Lambda = \begin{pmatrix} E_{n \times n} & 0 \\ 0 & \Lambda'_{m \times m} \end{pmatrix} = \left(\begin{array}{ccc|ccc} 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 & \cdots & 0 \\ \hline 0 & \cdots & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{array} \right), \quad (18)$$

where

$$\Lambda' = \begin{pmatrix} E_{(m-i) \times (m-i)} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{i \times i} \end{pmatrix}. \quad (19)$$

Now from the direct sum forms of Eqs.(17, 18), the task of classification under P_Λ, Q_Λ can be divided into two problems: the simultaneous similarity transformation of matrices γ_j and the construction of matrices β_j in canonical form. The first one can be solved with the help of results in Sec. 2.1. For the second problem the necessary and sufficient condition for the canonical form of β_j goes as follows:

$$r_{\max}(\Lambda' + \sum_{j=2}^L \alpha_j \beta_j) = m - i, \alpha_j \in \mathbb{C}. \quad (20)$$

Similar to the full rank situation, the transformation T here will also induce a symmetry \mathcal{P} to the canonical form, which can be schematically expressed as

$$\vec{\mathbf{b}}' = \mathcal{P} \vec{\mathbf{b}}, \quad (21)$$

where $\vec{\mathbf{b}} = (\lambda_i, a_{in}, \beta_{jkl})$.

In all one can conclude that for each matrix triple (E, J, A) , the following theorem holds:

Theorem 2.1 *The classification of $L \times N \times N$ can be represented in a triple-matrix form (E, J, A) up to a parametric symmetry \mathcal{P} .*

Now we note that the entanglement classification of quantum state $L \times N \times N$ under SLOCC can be decomposed into two problems: the simultaneous similarity transformation of a commuting matrix pair and the symmetry analysis of the corresponding canonical form. Therefore the difficulty in entanglement classification then turns to that in the symmetry property study, which requires further analysis for different entanglement structures. Analysis of the symmetry properties represents the essence of the entanglement class in our classification scheme. In the following we show the explicit form of symmetry in the classification of the $3 \times N \times N$ quantum system, from which the symmetry nature of state $L \times N \times N$ can be figured out.

3 Entanglement classification of $3 \times N \times N$ system

In this section we give an explicit example of a $3 \times N \times N$ system: $\psi = (A_1, A_2, A_3)$, where $r(A_1) = N$ (full-rank case), and when transformed into (E, A'_2, A'_3) via P, Q it has $[A'_2, A'_3] = 0$. From the discussion in Sec. 2 it is clear that the canonical form of this state is (E, J, A) , where J is the Jordan form of A'_2 and $[J, A] = 0$. We can always achieve $r(E) > r(J) > r(A)$ by applying a certain transformation T . In order to realize the unique entanglement classification, we need to find out the specific form of symmetry \mathcal{P} .

An invertible matrix T can be decomposed as $T = \mathbb{P}LDU$, where L (U) is the lower (upper) triangular with all diagonal entries equal to 1, D is a nonsingular diagonal matrix, and \mathbb{P} is a permutation matrix [9]. Among these four matrices \mathbb{P}, L, D, U : (1) \mathbb{P} and L are now contradicting with $r(E) > r(J) > r(A)$ in most cases, except for very limited (less than the number of distinct values of λ_i) and isolated values which can be treated individually; and (2) D corresponds to a trivial rescale of the whole matrix which can be treated by the renormalization of quantum state. Now the remaining effective T transformation is

$$U = \begin{pmatrix} 1 & z_1 & z_2 \\ 0 & 1 & z_3 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & z_1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & z_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & z_3 \\ 0 & 0 & 1 \end{pmatrix}. \quad (22)$$

The admissible T matrix which can be applied on the canonical form (E, J, A) is now

decomposed into a sequence of three operations, like

$$\begin{pmatrix} E \\ J \\ A \end{pmatrix} \xrightarrow{T_{[EJ]}^{(z_1)}} \begin{pmatrix} E + z_1 J \\ J \\ A \end{pmatrix}, \begin{pmatrix} E \\ J \\ A \end{pmatrix} \xrightarrow{T_{[EA]}^{(z_2)}} \begin{pmatrix} E + z_2 A \\ J \\ A \end{pmatrix}, \begin{pmatrix} E \\ J \\ A \end{pmatrix} \xrightarrow{T_{[JA]}^{(z_3)}} \begin{pmatrix} E \\ J + z_3 A \\ A \end{pmatrix}, \quad (23)$$

where $T_{[XY]}^{(z)}$ is an elementary operation that multiplies Y with z and adds it to the X column. Because all the E, J, A are direct sums of block-diagonalized submatrices and have the same partitions, we can therefore treat Eq.(23) with their diagonalized blocks.

3.1 The first superposition $T_{[EJ]}^{(z_1)}$

After the transformation $T_{[EJ]}^{(z_1)}$, the initial canonical form turns to

$$\begin{pmatrix} E + z_1 J \\ J \\ A \end{pmatrix}, \quad (24)$$

which is no longer a canonical form. To obtain the canonical form corresponding to it, operators P and Q should satisfy

$$P(E + z_1 J_{n_i}(\lambda_i))Q = E, \quad (25)$$

for each block. This equation can be reexpressed as

$$P g_{[EJ]}^{(z_1)}(J_{n_i}(0))Q = E, \quad (26)$$

where $g_{[EJ]}^{(z_1)}(x) = 1 + z_1 \lambda_i + z_1 x$. The reciprocal of polynomial function $g_{[EJ]}^{(z_1)}(x)$ can be readily obtained, that is (note $J_{n_i}^n(0) = 0$ if $n \geq n_i$),

$$\zeta_{[EJ]}^{(z_1)}(J_{n_i}(0)) = \left[g_{[EJ]}^{(z_1)}(J_{n_i}(0)) \right]^{-1} = \frac{1}{1 + z_1 \lambda_i} \sum_{n=0}^{n_i} \left(\frac{-z_1}{1 + z_1 \lambda_i} \right)^n J_{n_i}^n(0). \quad (27)$$

Since Eq.(26) leads to $P = Q^{-1} \cdot \zeta_{[EJ]}^{(z_1)}(J_{n_i}(0))$, we can further get

$$\begin{aligned} P J_{n_i}(\lambda_i) Q &= Q^{-1} \cdot \zeta_{[EJ]}^{(z_1)}(J_{n_i}(0)) \cdot J_{n_i}(\lambda_i) \cdot Q \\ &= Q^{-1} \cdot \left[\frac{\lambda_i}{1 + z_1 \lambda_i} + \sum_{n=1}^{n_i} \frac{1}{(1 + z_1 \lambda_i)^2} \left(\frac{-z_1}{1 + z_1 \lambda_i} \right)^{n-1} J_{n_i}^n(0) \right] \cdot Q \\ &= Q^{-1} \cdot f_{[(EJ),J]}^{(z_1)}(J_{n_i}(0)) \cdot Q. \end{aligned} \quad (28)$$

Here $f_{[(EJ)J]}^{(z_1)}(J_{n_i}(0))$ is defined to be $\zeta_{[EJ]}^{(z_1)}(J_{n_i}(0)) \cdot J_{n_i}(\lambda_i)$, which represents the influence of the superposition of E and $z_1 J$ on J after the operation P, Q . Similarly, we can obtain this operation influence on A_i , i.e.,

$$\begin{aligned}
PA_iQ &= P \cdot f_{[A_i]}(J_{n_i}(0)) \cdot Q \\
&= Q^{-1} \cdot \zeta_{[EJ]}^{(z_1)}(J_{n_i}(0)) \cdot f_{[A_i]}(J_{n_i}(0)) \cdot Q \\
&= Q^{-1} \cdot \left[\sum_{n=0}^{n_i} \frac{1}{1+z_1\lambda_i} \sum_{m=0}^n \left(\frac{-z_1}{1+z_1\lambda_i} \right)^m a_{in-m} J_{n_i}^n(0) \right] \cdot Q \\
&= Q^{-1} \cdot f_{[(EJ)A_i]}^{(z_1)}(J_{n_i}(0)) \cdot Q.
\end{aligned} \tag{29}$$

In order to see what the canonical form becomes now, we perform a similarity transformation S on the second matrix, Eq.(28), and transform it to the Jordan form

$$S^{-1}Q^{-1} \cdot f_{[(EJ)J]}^{(z_1)}(J_{n_i}(0)) \cdot QS = J_{n_i} \left(\frac{\lambda_i}{1+z_1\lambda_i} \right), \tag{30}$$

which can be verified with the help of (28). From (30) we then have

$$S^{-1}Q^{-1} \cdot J_{n_i}(0) \cdot QS = f_{[(EJ)J]}^{(z_1)-1} \left[J_{n_i} \left(\frac{\lambda_i}{1+z_1\lambda_i} \right) \right]. \tag{31}$$

Here $f_{[(EJ)J]}^{(z_1)-1}$ is the inverse function of $f_{[(EJ)J]}^{(z_1)}$. This similarity transformation will also make

$$\begin{aligned}
S^{-1}Q^{-1} f_{[(EJ)A_i]}^{(z_1)}(J_{n_i}(0))QS &= f_{[(EJ)A_i]}^{(z_1)}(S^{-1}Q^{-1}J_{n_i}(0)QS) \\
&= f_{[(EJ)A_i]}^{(z_1)} \circ f_{[(EJ)J]}^{(z_1)-1} \circ J_{n_i} \left(\frac{\lambda_i}{1+z_1\lambda_i} \right),
\end{aligned} \tag{32}$$

where the Jordan canonical form is treated as a polynomial function [see Eq.(12)] and $f \circ g(x) \equiv f(g(x))$.

We note that when the T transformation of $E + z_1 J$ applies on

$$(E, J_{n_i}(\lambda_i), A_i), \tag{33}$$

where $A_i = f_{[A_i]}(J_{n_i}(0))$, it will make the canonical form transform into

$$(E, J_{n_i} \left(\frac{\lambda_i}{1+z_1\lambda_i} \right), A'_i), \tag{34}$$

where $A'_i = f_{[(EJ)A]}^{(z_1)} \circ f_{[(EJ)J]}^{(z_1)-1} \circ J_{n_i} \left(\frac{\lambda_i}{1 + z_1 \lambda_i} \right)$. Though (33) and (34) are similar in block partitions with different parameters, they are equivalent entanglement classes. That means the entanglement classes parameterized by $\vec{a} = (\lambda_i, a_{in})$ are not inequivalent with different parametric values, while parameters undergo a transformation \mathcal{P}_{z_1} characterized by z_1 , i.e.,

$$\left. \begin{matrix} \lambda_i \\ a_{in} \end{matrix} \right\} \xrightarrow{\mathcal{P}_{z_1}} \left\{ \begin{matrix} \frac{\lambda_i}{1 + z_1 \lambda_i} \\ a'_{in} \end{matrix} \right. , \quad (35)$$

where a'_{in} is the entries of A'_i defined in Eq.(34), or simply,

$$\begin{pmatrix} E \\ J' \\ A' \end{pmatrix} = \mathcal{P}_{z_1} \begin{pmatrix} E \\ J \\ A \end{pmatrix} . \quad (36)$$

3.2 The second superposition $T_{[EA]}^{(z_2)}$

Similar to the above section, the transformation requirement

$$P(E + z_2 A_i)Q = E \quad (37)$$

leads to

$$P \cdot g_{[EA]}^{(z_2)} [J_{n_i}(0)] \cdot Q = E , \quad (38)$$

where the polynomial $g_{[EA]}^{(z_2)}(x) = \sum_{n=0} a_n x^n$, with $a_0 = 1 + z_2 a_{i0}$, $a_j = z_2 a_{ij}$ when $j > 0$. Define $\zeta_{[EA]}^{(z_2)}(J_{n_i}(0)) = [g_{[EA]}^{(z_2)}(J_{n_i}(0))]^{-1}$ such that $P = Q^{-1} \cdot \zeta_{[EA]}^{(z_2)}(J_{n_i}(0))$. What we need to know now are the values of $PJ_{n_i}(\lambda_i)Q$ and PA_iQ .

For $PJ_{n_i}(\lambda_i)Q$, we have

$$\begin{aligned} PJ_{n_i}(\lambda_i)Q &= Q^{-1} \cdot \zeta_{[EA]}^{(z_2)}(J_{n_i}(0)) \cdot J_{n_i}(\lambda_i) \cdot Q \\ &= Q^{-1} \cdot f_{[(EA)J]}^{(z_2)}(J_{n_i}(0)) \cdot Q . \end{aligned} \quad (39)$$

Here $f_{[EAJ]}^{(z_2)}(J_{n_i}(0)) \equiv \zeta_{[EJ]}^{(z_2)}(J_{n_i}(0)) \cdot J_{n_i}(\lambda_i)$. For PA_iQ , we have

$$\begin{aligned} PA_iQ &= P \cdot f_{[A_i]}(J_{n_i}(0)) \cdot Q \\ &= Q^{-1} \cdot \zeta_{[EA]}^{(z_2)}(J_{n_i}(0)) \cdot f_{[A_i]}(J_{n_i}(0)) \cdot Q \\ &= Q^{-1} \cdot f_{[(EA)A]}^{(z_2)}(J_{n_i}(0)) \cdot Q. \end{aligned} \quad (40)$$

There always exists a similarity transformation such that

$$S^{-1}Q^{-1}f_{[(EA)J]}^{(z_2)}(J_{n_i}(0))QS = J_{n_i} \left(\frac{\lambda_i}{1 + z_2 a_{i0}} \right) \quad (41)$$

and hence

$$S^{-1}Q^{-1}J_{n_i}(0)QS = f_{[(EA)J]}^{(z_2)-1} \left[J_{n_i} \left(\frac{\lambda_i}{1 + z_2 a_{i0}} \right) \right]. \quad (42)$$

Then we have

$$\begin{aligned} &S^{-1}Q^{-1}f_{[(EA)A]}^{(z_2)}(J_{n_i}(0))QS \\ &= f_{[(EA)A]}^{(z_2)}(S^{-1}Q^{-1}J_{n_i}(0)QS) \\ &= f_{[(EA)A]}^{(z_2)} \circ f_{[(EA)J]}^{(z_2)-1} \circ J_{n_i} \left(\frac{\lambda_i}{1 + z_2 a_{i0}} \right). \end{aligned} \quad (43)$$

We note that when the T transformation of $E + z_2 A_i$ applies on

$$(E, J_{n_i}(\lambda_i), A_i), \quad (44)$$

where $A_i = f_{[A_i]}(J_{n_i}(0))$, it will make the canonical form transform into

$$(E, J_{n_i} \left(\frac{\lambda_i}{1 + z_2 a_{i0}} \right), A_i''), \quad (45)$$

where $A_i'' = f_{[(EA)A]}^{(z_2)} \circ f_{[(EA)J]}^{(z_2)-1} \circ J_{n_i} \left(\frac{\lambda_i}{1 + z_2 a_{i0}} \right)$. The entanglement classes parameterized by $\vec{a} = (\lambda_i, a_{in})$ are not inequivalent with different parametric values, while the parameters undergo a transformation \mathcal{P}_{z_2} characterized by z_2 , i.e.,

$$\left. \begin{matrix} \lambda_i \\ a_{in} \end{matrix} \right\} \xrightarrow{\mathcal{P}_{z_2}} \left\{ \begin{matrix} \frac{\lambda_i}{1 + z_2 a_{i0}} \\ a_{in}'' \end{matrix} \right. . \quad (46)$$

Here, a_{in}'' are elements of A_i'' .

3.3 The third superposition $T_{[JA]}^{(z_3)}$

In this case

$$J_{n_i} + z_3 A_i = g_{[JA]}^{(z_3)}(J_{n_i}(0)). \quad (47)$$

From the above equation it can be inferred that there should exist a similarity transformation

$$S^{-1} g_{[JA]}^{(z_3)}[J_{n_i}(0)] S = J_{n_i}(\lambda + z_3 a_{i0}) . \quad (48)$$

Then

$$S^{-1} J_{n_i}(0) S = g_{[JA]}^{(z_3)-1}[J_{n_i}(\lambda + z_3 a_{i0})] \quad (49)$$

and

$$\begin{aligned} S^{-1} A_i S &= S^{-1} \cdot f_{[A_i]}(J_{n_i}(0)) \cdot S \\ &= f_{[A_i]} \circ g_{[JA]}^{(z_3)-1} \circ J_{n_i}(\lambda + z_3 a_{i0}) . \end{aligned} \quad (50)$$

We note that when the T transformation of $J_{n_i} + z_3 A_i$ applies on

$$(E, J_{n_i}(\lambda_i), A_i) , \quad (51)$$

where $A_i = f_{[A_i]}(J_{n_i}(0))$, it will make the canonical form transform into

$$(E, J_{n_i}(\lambda_i + z_3 a_{i0}), A_i''') , \quad (52)$$

where $A_i''' = f_{[A_i]} \circ g_{[JA]}^{(z_3)-1} \circ J_{n_i}(\lambda + z_3 a_{i0})$. The entanglement classes parameterized by $\vec{\mathbf{a}} = (\lambda_i, a_{in})$ are not completely inequivalent with different parametric values, while parameters undergo a transformation \mathcal{P}_{z_3} characterized by z_3 , i.e.,

$$\left. \begin{array}{c} \lambda_i \\ a_{in} \end{array} \right\} \xrightarrow{\mathcal{P}_{z_3}} \left\{ \begin{array}{c} \lambda_i + z_3 a_{i0} \\ a_{in}''' \end{array} \right. . \quad (53)$$

Here, a_{in}''' are elements of A_i''' .

3.4 Results

As far we know that different A are inequivalent up to the following transformations

$$A'_i = f_{[(EJ)A]}^{(z_1)} \circ f_{[(EJ)J]}^{(z_1)-1} \circ J_{n_i} \left(\frac{\lambda_i}{1 + z_1 \lambda_i} \right), \quad (54)$$

$$A'_i = f_{[(EA)A]}^{(z_2)} \circ f_{[(EA)J]}^{(z_2)-1} \circ J_{n_i} \left(\frac{\lambda_i}{1 + z_2 a_{i0}} \right), \quad (55)$$

$$A'_i = f_{[A_i]} \circ g_{[JA]}^{(z_3)-1} \circ J_{n_i} (\lambda + z_3 a_{i0}). \quad (56)$$

or their combinations in proper orders.

$$\begin{pmatrix} E \\ J' \\ A' \end{pmatrix} = \mathcal{P} \begin{pmatrix} E \\ J \\ A \end{pmatrix}, \quad (57)$$

where $\mathcal{P} = \prod_z \mathcal{P}_z$, $A = \oplus A_i$.

Notice that the above-explained classification scheme should be applicable to the $2 \times N \times N$ case, a special case of the general $L \times N \times N$ system. By then there will be only the superposition $T_{[EJ]}^{(z_1)}$, and (E, J) is just the entanglement class up to the symmetry [3]

$$\lambda_i \xrightarrow{\mathcal{P}_{z_1}} \frac{\lambda_i}{1 + z_1 \lambda_i}, \quad (58)$$

which is a special case of Eq.(35).

3.5 Examples

In the following we employ the new method to characterize the entanglement classes of $2 \times N \times N$ and $3 \times N \times N$ state under SLOCC. First, we show how the present method works by comparing it with the previous result [4] for a $2 \times N \times N$ system (Appendix A in [4]); then we use the present method to construct the canonical form for the specific $3 \times N \times N$ system discussed above.

3.5.1 $2 \times N \times N$ case

The quantum state of $2 \times N \times N$ can be represented by a matrix pair $\Psi = (\Gamma_1, \Gamma_2)$. The invertible operations (SLOCC operations) act on the three partite in the following

forms:

$$\Psi' = T \otimes P \otimes Q \Psi = T \begin{pmatrix} P\Gamma_1 Q \\ P\Gamma_2 Q \end{pmatrix}, \quad (59)$$

where T is a 2×2 matrix. Suppose one typical canonical form of an entanglement class is

$$\Psi = (E, J) = \left(\begin{pmatrix} \ddots & & & \\ & 1 & 0 & 0 \\ & 0 & 1 & 0 \\ & 0 & 0 & 1 \\ & & & \ddots \end{pmatrix}, \begin{pmatrix} \ddots & & & \\ & \lambda_i & 1 & 0 \\ & 0 & \lambda_i & 1 \\ & 0 & 0 & \lambda_i \\ & & & \ddots \end{pmatrix} \right). \quad (60)$$

Because of the requirement $r(E) > r(J)$, the admissible T matrix that keeps the rank inequality should be in the form of an upper triangular $\begin{pmatrix} t_{11} & t_{12} \\ 0 & t_{22} \end{pmatrix}$ [3]. For the sake of convenience we take the pair (Γ_1, Γ_2) to represent the two 3×3 submatrices hereafter. After the T transformation

$$\Gamma'_1 = \begin{pmatrix} t_{11} + t_{12}\lambda_i & t_{12} & 0 \\ 0 & t_{11} + t_{12}\lambda_i & t_{12} \\ 0 & 0 & t_{11} + t_{12}\lambda_i \end{pmatrix}, \quad \Gamma'_2 = \begin{pmatrix} t_{22}\lambda_i & t_{22} & 0 \\ 0 & t_{22}\lambda_i & t_{22} \\ 0 & 0 & t_{22}\lambda_i \end{pmatrix}, \quad (61)$$

the canonical form corresponding to it is now (i.e., $P = Q^{-1}\Gamma_1'^{-1}$)

$$\begin{aligned} \Psi' &= (P\Gamma'_1 Q, P\Gamma'_2 Q) = (E, J') \\ &= \left(\begin{pmatrix} \ddots & & & \\ & 1 & 0 & 0 \\ & 0 & 1 & 0 \\ & 0 & 0 & 1 \\ & & & \ddots \end{pmatrix}, \begin{pmatrix} \ddots & & & \\ & \frac{t_{22}\lambda_i}{t_{11}+t_{12}\lambda_i} & 1 & 0 \\ & 0 & \frac{t_{22}\lambda_i}{t_{11}+t_{12}\lambda_i} & 1 \\ & 0 & 0 & \frac{t_{22}\lambda_i}{t_{11}+t_{12}\lambda_i} \\ & & & \ddots \end{pmatrix} \right). \end{aligned} \quad (62)$$

One can regard this as a symmetry among parameters $\lambda_i \sim \lambda'_i = \frac{t_{22}\lambda_i}{t_{11}+t_{12}\lambda_i}$ induced by the transformation T in canonical forms. In the Appendix A of [4], this symmetry is expressed as

$$(E, J) \xrightarrow{T, P, Q} (E, J'), \quad (63)$$

where the parameters in (E, J') are $\lambda'_i = \frac{t_{22}\lambda_i}{t_{11}+t_{12}\lambda_i}$ and thus the canonical forms with different values of the parameter actually belong to the same SLOCC class, that is, $\lambda_i \sim \lambda'_i$.

By employing the method developed in this work, the quantum state can be manipulated as follows:

$$\Psi = (\Gamma_1, \Gamma_2) = (1, \lambda_i + x) , \quad (64)$$

where x represents $J_{n_i}(0)$. The transformation T makes

$$(\Gamma'_1, \Gamma'_2) = (t_{11} + t_{12}\lambda_i + t_{12}x , t_{22}\lambda_i + t_{22}x) . \quad (65)$$

The canonical form now can be obtained just by dividing the polynomial $t_{11} + t_{12}\lambda_i + t_{12}x$ [the same role as $\Gamma_1'^{-1}$ before Eq.(62)]:

$$(P\Gamma'_1Q, P\Gamma'_2Q) = \left(1 , \frac{t_{22}\lambda_i + t_{22}x}{t_{11} + t_{12}\lambda_i + t_{12}x} \right) . \quad (66)$$

Here $P\Gamma'_2Q$ can be expanded as

$$\frac{t_{22}\lambda_i + t_{22}x}{t_{11} + t_{12}\lambda_i + t_{12}x} = \frac{t_{22}\lambda_i}{t_{11} + t_{12}\lambda_i} + \frac{t_{11}t_{22}}{(t_{11} + t_{12}\lambda_i)^2}x - \frac{t_{11}t_{12}t_{22}}{(t_{11} + t_{12}\lambda_i)^3}x^2 . \quad (67)$$

Note that here $J_{n_i}(0)^3 = 0$. The the Jordan form of the second matrix in Eq.(66) is $\frac{t_{22}\lambda_i}{t_{11} + t_{12}\lambda_i} + J_{n_i}(0)$, and the canonical form is then

$$\Psi' = (1 , \lambda'_i + x) = \left(1 , \frac{t_{22}\lambda_i}{t_{11} + t_{12}\lambda_i} + x \right) . \quad (68)$$

Compared to Eq.(64) we see that different parameters λ_i and $\lambda'_i = \frac{t_{22}\lambda_i}{t_{11} + t_{12}\lambda_i}$ belong to the same entanglement class induced by the transformation T , which is in agreement with the result of Eq.(63).

3.5.2 $3 \times N \times N$ case

We consider the case of $\psi = (A_1, A_2, A_3)$ where $r(A_1) = N$. Here, the ψ can always be transformed into $\psi' = (E, A'_2, A'_3)$ by invertible operators P, Q . The actual problem discussed in Sec. 3 is to construct the canonical form of (E, A'_2, A'_3) under T, P, Q , where $[A'_2, A'_3] = 0$. With our method, this can be decomposed into two tasks: (1) the simultaneous similarity transformation of matrix pairs (A'_2, A'_3) (canonical form) and (2) application of superpositions induced by T among (E, A'_2, A'_3) (symmetry between the canonical form).

In the triple-matrix form (E, A_2, A_3) , its canonical form under P, Q is (E, J, A) . The effective T transforms (E, J, A) into $(E + z_1 J + z_2 A, J + z_3 A, A)$ whose canonical form is (E, J', A') . That is, T induces a symmetry (equivalent relation) between different canonical forms (E, J, A) and (E, J', A') . Our task is to find the relation between parameters in J', A' and J, A . The initial canonical form can be represented by a polynomial form, i.e.,

$$\begin{aligned}
\psi &= (E, J, A) \\
&= \left(\begin{pmatrix} \ddots & & & \\ & 1 & 0 & 0 \\ & 0 & 1 & 0 \\ & 0 & 0 & 1 \\ & & & \ddots \end{pmatrix}, \begin{pmatrix} \ddots & & & \\ & \lambda_i & 1 & 0 \\ & 0 & \lambda_i & 1 \\ & 0 & 0 & \lambda_i \\ & & & \ddots \end{pmatrix}, \begin{pmatrix} \ddots & & & \\ & a_{i0} & a_{i1} & a_{i2} \\ & 0 & a_{i0} & a_{i1} \\ & 0 & 0 & a_{i0} \\ & & & \ddots \end{pmatrix} \right) \\
&= (1, \lambda_i + x, a_{i0} + a_{i1}x + a_{i2}x^2), \tag{69}
\end{aligned}$$

where x represents $J_{n_i}(0)$.

In the following we show that the transformation $T_{[EJ]}^{(z_1)}$ in Sec. 3.1 may transform the above quantum state into the following form:

$$(1 + z_1 \lambda_i + z_1 x, \lambda_i + x, a_{i0} + a_{i1}x + a_{i2}x^2). \tag{70}$$

We transform the first matrix into a unit matrix by dividing the polynomial $1 + z_1 \lambda_i + z_1 x$ and obtain the polynomial functions $f_{[(EJ)J]}^{(z_1)}(x)$ and $f_{[(EJ)A]}^{(z_1)}(x)$. That is,

$$\begin{aligned}
&\left(1, \frac{\lambda_i + x}{1 + z_1 \lambda_i + z_1 x}, \frac{a_{i0} + a_{i1}x + a_{i2}x^2}{1 + z_1 \lambda_i + z_1 x} \right) \\
&= \left(1, \frac{\lambda_i}{1 + z_1 \lambda_i} + \frac{x}{(1 + z_1 \lambda_i)^2} - \frac{z_1 x^2}{(1 + z_1 \lambda_i)^3}, \right. \\
&\quad \frac{a_{i0}}{1 + z_1 \lambda_i} + \frac{(a_{i1} - a_{i0}z_1 + a_{i1}z_1 \lambda_i)x}{(1 + z_1 \lambda_i)^2} + \\
&\quad \left. \frac{a_{i2}(1 + z_1 \lambda_i)^2 - z_1(a_{i1} - a_{i0}z_1 + a_{i1}z_1 \lambda_i)}{(1 + z_1 \lambda_i)^3} x^2 \right) \\
&= \left(1, f_{[(EJ)J]}^{(z_1)}(x), f_{[(EJ)A]}^{(z_1)}(x) \right), \tag{71}
\end{aligned}$$

where only up to x^2 is needed due to $J_{n_i}^3(0) = 0$. And, then we transform the second matrix into Jordan form, which means the similarity transformation

$$M f_{[(EJ)J]}^{(z_1)}(x) M^{-1} = J_{n_i} \left(\frac{\lambda_i}{1 + z_1 \lambda_i} \right). \tag{72}$$

In polynomial form it reads

$$\frac{\lambda_i}{1+z_1\lambda_i} + \frac{y}{(1+z_1\lambda_i)^2} - \frac{z_1y^2}{(1+z_1\lambda_i)^3} = J_{n_i}\left(\frac{\lambda_i}{1+z_1\lambda_i}\right) = \frac{\lambda_i}{1+z_1\lambda_i} + x, \quad (73)$$

with $y = MxM^{-1}$, where $x = 0$ implies $y = 0$. Hence, the quantum state is now in the form

$$\left(1, J_{n_i}\left(\frac{\lambda_i}{1+z_1\lambda_i}\right), f_{[(EJ)A]}^{(z_1)}(y)\right) \quad (74)$$

correspondingly. One can solve y from Eq.(73) by expressing $y = \sum_{i=1}^{\infty} b_i x^i$, like

$$f_{[(EJ)J]}^{(z_1)-1}\left(J_{n_i}\left(\frac{\lambda_i}{1+z_1\lambda_i}\right)\right) = y = (1+z_1\lambda_i)^2 x + z_1(1+z_1\lambda_i)^3 x^2 + \dots. \quad (75)$$

Take y into $f_{[(EJ)A]}^{(z_1)}(y)$ and keep those terms up to x^2 , for the third matrix A'_i we have

$$\begin{aligned} A'_i &= a'_{i0} + a'_{i1}x + a'_{i2}x^2 \\ &= \frac{a_{i0}}{1+z_1\lambda_i} + (a_{i1} - a_{i0}z_1 + a_{i1}z_1\lambda_i)x + a_{i2}(1+z_1\lambda_i)^3 x^2. \end{aligned} \quad (76)$$

Finally, by comparing coefficients of different powers of x , we have

$$\begin{aligned} (\lambda_i, a_{i0}, a_{i1}, a_{i2}) &\sim (\lambda'_i, a'_{i0}, a'_{i1}, a'_{i2}) \\ &= \left(\frac{\lambda_i}{1+z_1\lambda_i}, \frac{a_{i0}}{1+z_1\lambda_i}, (a_{i1} - a_{i0}z_1 + a_{i1}z_1\lambda_i), a_{i2}(1+z_1\lambda_i)^3\right). \end{aligned} \quad (77)$$

Similarly, we can get the result for $T_{[EA]}^{(z_2)}$, i.e.,

$$\begin{aligned} (\lambda_i, a_{i0}, a_{i1}, a_{i2}) &\sim (\lambda'_i, a'_{i0}, a'_{i1}, a'_{i2}) \\ &= \left(\frac{\lambda_i}{1+z_2a_{i0}}, \frac{a_{i0}}{1+z_2a_{i0}}, \frac{a_{i1}}{1+a_{i0}z_2 - a_{i1}z_2\lambda_i}, \frac{a_{i2}(1+a_{i0}z_2)^3}{(1+z_2(a_{i0} - a_{i1}\lambda_i))^3}\right), \end{aligned} \quad (78)$$

and $T_{[JA]}^{(z_3)}$

$$\begin{aligned} (\lambda_i, a_{i0}, a_{i1}, a_{i2}) &\sim (\lambda'_i, a'_{i0}, a'_{i1}, a'_{i2}) \\ &= \left(\lambda_i + z_3a_{i0}, a_{i0}, \frac{a_{i1}}{1+a_{i1}z_3}, \frac{a_{i2}}{(1+z_3a_{i1})^3}\right). \end{aligned} \quad (79)$$

More specifically, take Eq.(79) as an example. If a quantum state in the canonical form has one block with parameters $(\lambda_i, a_{i0}, a_{i1}, a_{i2}) = (1, 0, 2, 3)$, then it is SLOCC equivalent to the canonical form which has the corresponding block of $(\lambda'_i, a'_{i0}, a'_{i1}, a'_{i2}) =$

$(1, 0, \frac{1}{1+2 \cdot z_3}, \frac{3}{(1+2 \cdot z_3)^3})$. In order to complete the classification, in some special cases, i.e., when rescaling the whole matrix of J or A as discussed before Eq.(22), zero appears in denominators in the fractions of Eqs.(78) and (79), and the states have to be treated separately. In all these cases, the method of similarity transformation plus the polynomial form of symmetries shall be applied iteratively.

4 Conclusion

In this work, we propose a general classification scheme for entangled quantum system $L \times N \times N$. In this scenario, based on the commutativity of the ILOs T , P , Q , the classification procedure is decomposed into two steps: the simultaneous similarity transformation of a commuting matrix pair into a canonical form and application of the internal symmetry of parameters in the canonical form. This is innovative to the general entanglement classifications, because by this scheme we can always extract symmetry properties from the general equivalent relation and then leave a relative simple canonical form for the entanglement class. In most cases the major challenge comes from the detailed forms of the representation of symmetries. For demonstration, a concrete example of entanglement classification for a type of $3 \times N \times N$ system is presented, of which the symmetry properties are expressed in forms of polynomial functions.

Acknowledgments

This work was supported in part by the National Natural Science Foundation of China(NSFC) and by the CAS Key Projects KJCX2-yw-N29 and H92A0200S2. We thank Bin Liu for initial collaboration on this work.

Appendix

A The structure of matrix A

I. If the Jordan form $J = \oplus J_{n_i}(\lambda_i)$ is nonderogatory, that is, every λ_i has geometric multiplicity 1, then matrices commuting with it can be expressed as $A = \oplus_i A_i$, where A_i is in the form of Eq.(11) (see, for example, theorem **S2.2** in [10]).

II. If some of the Jordan blocks of J have the same eigenvalue,

$$J = \cdots \oplus \underbrace{J_{n_1}(\lambda_1) \oplus J_{n_2}(\lambda_2) \oplus \cdots \oplus J_{n_l}(\lambda_l)}_{\lambda_1=\lambda_2=\cdots=\lambda_l} \oplus \cdots, \quad (80)$$

where $n_i \geq n_j$ for $i > j$. (We can always gather the Jordan blocks with same λ_i together.)

The matrix commuting with this block can be expressed as

$$A' = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1l} \\ A_{21} & A_{22} & \cdots & A_{2l} \\ \vdots & \vdots & \ddots & \vdots \\ A_{l1} & A_{l2} & \cdots & A_{ll} \end{pmatrix},$$

where A_{ij} is an $n_i \times n_j$ upper triangular Toeplitz matrix. As the main subject of our work is the study of parametric symmetries, we refer to Eqs. (S2.13–S2.15) in [10] for the detailed definition of A_{ij} . Here we give an example of $n_1 = 3, n_2 = 2$:

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} a_{110} & a_{111} & a_{112} & a_{120} & a_{121} \\ 0 & a_{110} & a_{111} & 0 & a_{120} \\ 0 & 0 & a_{110} & 0 & 0 \\ 0 & a_{210} & a_{211} & a_{220} & a_{221} \\ 0 & 0 & a_{210} & 0 & a_{220} \end{pmatrix}. \quad (81)$$

This can be represented phenomenologically by the following matrix of polynomials:

$$\begin{pmatrix} f_{[A_{11}]}(x) & f_{[A_{12}]}(x) \\ f_{[A_{21}]}(x) & f_{[A_{22}]}(x) \end{pmatrix} = \begin{pmatrix} a_{110} + a_{111}x + a_{112}x^2 & a_{120} + a_{121}x \\ a_{210}x + a_{211}x^2 & a_{220} + a_{221}x \end{pmatrix}, \quad (82)$$

where $x = J_{n_1}(0)$ given that $n_1 \geq n_2$, and the last $n_1 - n_2$ rows and columns of Eq.(82) should be omitted to give the form of Eq.(81). In the discussion of the parametric symmetry of the canonical form in Secs. 3.2 and 3.3, the supposition and inverse functions applied in Eq.(82) can be proceeded directly with the only difference being the emergence of the nondiagonal elements, i.e., $f_{[A_{12}]}$ and $f_{[A_{21}]}$ in Eq.(82).

References

- [1] M.A. Nielsen and I.L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, England, 2000).
- [2] W. Dür, G. Vidal, and J.I. Cirac, Phys. Rev. A **62**, 062314 (2000).
- [3] Shuo Cheng, Junli Li, Cong-Feng Qiao, J. Phys. A **43**, 055303 (2010).
- [4] Xikun Li, Junli Li, Bin Liu, and Cong-Feng Qiao, Sci. China G **54**, 1471 (2011).
- [5] G. R. Belitskii and V. V. Sergeichuk, Linear Algebra and its Application **361**, 203 (2003).
- [6] Bin Liu, Jun-Li Li, Xikun Li, and Cong-Feng Qiao, Phys. Rev. Lett. **108**, 050501 (2012).
- [7] I. M. Gel'fand and V. A. Ponomarev, Translated from Funktsional'nyi Analiz i Ego Prilozheniya, **3**, pp. 81-82 (1969).
- [8] V. M. Bondarenko, T. G. Gerasimova, and V. V. Sergeichuk, Linear Algebra and its Applications, **430**, 86 (2009).
- [9] R. A. Horn and C. R. Johnson, *Matrix Analysis*, (Cambridge University, Cambridge England, 1985).
- [10] I. Gohberg, P. Lancaster, and L. Rodman, *Matrix Polynomials* (Academic Press, New York, 1982).